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SOME ESTIMATION METHODS FOR THE PARAMETER OF SUJATHA DISTRIBUTION: A MONTE CARLO SIMULATION

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ABSTRACT

This paper investigates and compares six parameter estimation procedures for the one-parameter Sujatha distribution originally proposed by Shanker (2016). Specifically, we examine the Maximum Likelihood Estimator (MLE), the Method of Moments (MOM), the Modified Method of Moments (MMOM), and three percentile estimators ($PE_{0.25}$, $PE_{0.50}$, and $PE_{0.75}$) through an extensive Monte Carlo simulation study comprising 5,000 replications. Performance is evaluated using Mean Square Error (MSE) across sample sizes $n \in \{20, 30, 50, 100\}$ and parameter values $\theta \in \{1, 1.5, 2, 3\}$. Results indicate that MOM is consistently the most efficient estimator for $\theta \leq 2$ with larger sample sizes, while MLE performs best for small samples ($n = 20$) and modest parameter values. The MMOM provides reliable moderate MSE performance for small to medium samples. Percentile estimators are generally less efficient, though $PE_{0.50}$ becomes competitive for $\theta = 3$ and large n. Overall, MOM is the generally recommended estimator for $\theta \leq 2$, while $PE_{0.75}$ is preferred for large θ and large sample sizes. These findings provide practical guidelines for selecting estimation methods when working with the Sujatha distribution in lifetime data analysis

Keywords: Sujatha distribution, parameter estimation, maximum likelihood, Monte Carlo simulation, mean square error.

1. Introduction

Lifetime distributions play a pivotal role in reliability engineering, survival analysis, and the modelling of real-world phenomena characterized by non-negative random variables. Among recently proposed one-parameter distributions, the Sujatha distribution, introduced by Shanker (2016), has attracted considerable attention due to its increasing hazard rate and its ability to outperform competing models such as Akash, Lindley, Shanker, and Exponential distributions on several empirical datasets.

The probability density function (PDF) of the Sujatha distribution is given by:

$$f(x;\theta) = \frac{\theta^3}{\theta^2+\theta+12}\left(x^2+x+1\right)e^{-\theta x}, x>0, \theta>0 \quad (1)$$

and its cumulative distribution function (CDF) is:

$$F(x;\theta)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}, x>0, \theta>0 \quad (2)$$

In equations (1) and (2), x > 0 is the random variable representing the lifetime or failure time, and θ > 0 is the single unknown scale parameter to be estimated from observed data. The parameter θ controls the rate of decay of the distribution: smaller values of θ yield a slower exponential decay producing a longer right tail and a more spread-out, heavy right-skewed density, while larger values of θ produce a faster exponential decay with probability mass more concentrated near zero.

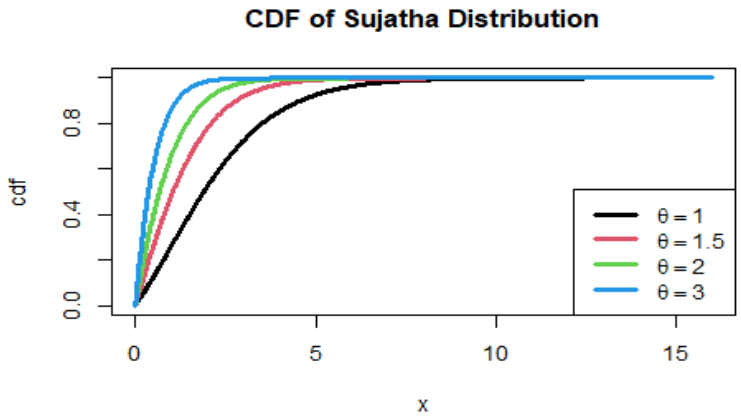


Figure 1: Cumulative Distribution Function (CDF) of the Sujatha Distribution with actual parameter values

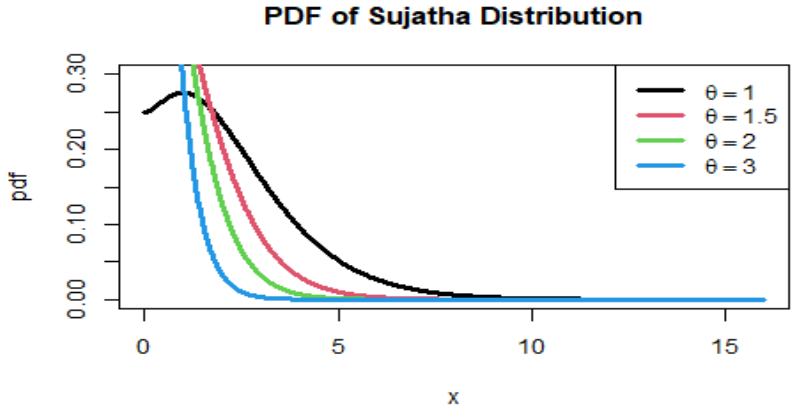


Figure 2: Probability Distribution Function (PDF) of the Sujatha Distribution with actual parameter values

Shanker (2016) employed the Maximum Likelihood Estimator (MLE) and the Method of Moments (MOM) for parameter estimation. Subsequent work by Nwikpe and Iwok (2020) extended the distribution to a three-parameter variant to enhance flexibility. Despite these contributions, a systematic comparative study of estimation procedures for the original one-parameter Sujatha distribution remains limited in the literature. It is well established that MLE may exhibit significant bias for small samples under a frequentist framework. Alternative estimators such as percentile-based estimators and modified moment estimators have been shown in other distributional contexts to mitigate these shortcomings (Zaka & Akhter, 2014; Bhatti et al., 2018).

This paper therefore addresses the following question: among the MLE, MOM, MMOM, $PE_{0.25}$, $PE_{0.50}$, and $PE_{0.70}$, which estimator is most efficient for the Sujatha distribution under varying sample sizes and parameter settings? It is important to note that the relative performance of these estimation methods depends critically on the prevailing conditions. MLE, for instance, is known to be asymptotically efficient and consistent, but may underperform in small samples due to violation of asymptotic regularity conditions. MOM estimators are generally simpler and more robust in small samples but can be less efficient than MLE as sample size grows. Modified moment estimators exploit additional distributional characteristics such as the coefficient of variation to improve on classical MOM. Percentile estimators, which match empirical and theoretical quantiles, tend to be competitive when the distributional tails are well-behaved and the chosen quantile aligns with regions of high information content. The performance of all these estimators also depends on the structural shape of the distribution, which in the Sujatha distribution is governed entirely by θ. The remainder of the paper is organized as follows. Section 1.1 reviews related literature. Section 2 derives each estimator analytically and describes the Monte Carlo simulation design. Section 3 presents and interprets the MSE results. Section 4 concludes with recommendations.

1.1 Literature Review

Parameter estimation for lifetime and reliability distributions has been extensively studied. George and Florence (2014) conducted a simulation study comparing shape and scale estimators for the two-parameter Weibull distribution, finding that MLE outperforms competing methods. Similarly, Neil B. Marks (2005) examined percentile estimators for Weibull parameters and found that PE based on the 10th and 90th percentiles is marginally inferior to MLE in variability.

Zaka and Akhter (2014) estimated parameters of the power function distribution using modified moments, MLE, and percentile estimators, demonstrating that certain modified estimators outperform their classical counterparts in terms of bias and MSE. Bhatti et al. (2018) proposed modified percentile estimators for the Pareto distribution, showing that estimators based on the expectation of the empirical CDF of the first-order statistic yield more precise estimates than MLE and moment-based approaches.

Francisco et al. (2016) evaluated several estimation procedures for the Extended Exponential Geometric distribution and found that the maximum product of spacings estimator achieves the lowest MSE. Owen (2008) compared PERT, TSP, and quantile estimators for the Beta distribution, concluding that quantile estimators perform comparably to MLE and MOM across most parameter settings.

Mert Kantar (2016) proposed weighted least squares estimators for log-logistic and Weibull distributions, showing superior MSE performance relative to MLE and ordinary least squares. Rafiq (1996) demonstrated that fractional moment estimation for the Gamma distribution is nearly equivalent to MLE in terms of asymptotic variance, while substantially outperforming the classical method of moments.

Yenilmez et al. (2018) studied censored regression models with non-normal errors, confirming the consistency and efficiency of MLE in that context. Collectively, these studies motivate a rigorous comparison of estimation procedures specifically tailored to the Sujatha distribution, which has not been undertaken in the literature. The relevance of the distributions reviewed above to the Sujatha distribution lies in their shared structural characteristics as one-parameter or few-parameter lifetime distributions with non-negative support and right-skewed density shapes. Like the Sujatha distribution, the Lindley, Akash, and exponential-family distributions exhibit increasing or unimodal hazard rates, making insights from their estimation literature directly applicable. Specifically, findings that MLE can be biased in small samples,

that modified moment estimators can outperform classical MOM, and that percentile estimators are competitive in heavy-tailed settings all carry direct implications for the choice of estimation method for the Sujatha distribution. The present study systematically evaluates these implications through Monte Carlo simulation.

2. Methodology

2.1 Maximum Likelihood Estimation (MLE)

Given an independent random sample $x_1, x_2, \ldots, x_n$ from the Sujatha distribution, the log-likelihood function is:

$$\ln\left(\ell\left(\theta\right)\right) = 3n\left(\ln\theta\right) - n\left(\ln\left(\theta^2+\theta+2\right)\right) + n\left(\ln\left(x_i^2+x_i+1\right)\right) - \theta\sum_{i=1}^{n} x_i \tag{3}$$

Differentiating $\ell(\theta)$ with respect to θ and equating to zero yields the score equation:

$$\frac{\partial \ell}{\partial \theta} = \frac{3n}{\theta} - \frac{n\left(2\theta+1\right)}{\theta^2+\theta+2} - n\bar{x} \tag{4}$$

Simplifying equation (4), the MLE $\hat{\theta}$ satisfies the cubic equation:

$$\bar{x}\hat{\theta}^3 + \hat{\theta}^2\left(\bar{x}-1\right) + 2\hat{\theta}\left(\bar{x}-1\right) - 6 = 0 \tag{5}$$

This equation is solved numerically via Newton–Raphson iteration to obtain the optimal solution.

2.2 Method of Moments (MOM)

Equating the theoretical first moment of the Sujatha distribution to the sample mean $\bar{x}$, the population mean evaluates to:

$$E\left(X\right) = \int_0^{\infty} x f\left(x;\theta\right)dx = \frac{\theta}{\theta^2+\theta+2}\left\{\frac{6}{\theta^2}+\frac{2}{\theta}+1\right\} \tag{6}$$

$$\bar{x} = \frac{\sum_{i-1}^{n} x_i}{n} \tag{7}$$

Setting $E(X) = \bar{x}$ and simplifying yields the same cubic as equation (5), demonstrating that for the Sujatha distribution the MLE and MOM produce identical estimating equations, a notable theoretical result reported by Shanker (2016).

2.3 Modified Method of Moments (MMOM)

The Modified Method of Moments exploits the coefficient of variation (CV) of the Sujatha distribution:

$$CV = \frac{\sqrt{\theta^4+4\theta^3+18\theta^2+12\theta+12}}{\theta^2+2\theta+6} \tag{8}$$

Setting CV = s / $\bar{x}$ (where s is the sample standard deviation), squaring, and rearranging yields the fourth-degree polynomial estimating equation:

$$\left(\bar{X}^2-S^2\right)\hat{\theta}^4 + 4\left(\bar{X}^2-S^2\right)\hat{\theta}^3 + 2\left(\bar{X}^2-S^2\right)\hat{\theta}^2 + 12\left(\bar{X}^2-S^2\right)\hat{\theta} + 12\left(\bar{X}^2-S^2\right) = 0 \tag{9}$$

The real positive root of equation (9) provides $\hat{\theta}$ MMOM.

2.4 Percentile Estimators (PE)

Percentile estimators are obtained by matching theoretical quantiles of the Sujatha distribution to their sample counterparts. Let p_n denote the n^{th} sample percentile. Substituting $F(p_n; \theta) = \pi_n$ into the CDF (equation 2) and differentiating with respect to θ gives the general estimating equation:

$$\hat{\theta}^2\left(\Delta_1-\Delta_2+4\Delta_1\Delta_3\right)+2\hat{\theta}\left(2\Delta_1+\Delta_1\Delta_3+\Delta_2\Delta_3-2\right)+\left(2\Delta_2+\Delta_2\Delta_3-2\right)=0 \quad (10)$$

where $\Delta_1 = p_n^2 + p_n + 1$, $\Delta_2 = 2p_n + 1$, and $\Delta_3 = p_n$.

Three specific percentile estimators are considered: $PE_{0.25}$ (first quartile), $PE_{0.50}$ (median), and $PE_{0.75}$ (third quatile), obtained by substituting p_{25}, p_{50}, and p_{75} into equation (10), respectively.

2.5 Monte Carlo Simulation Design

A Monte Carlo simulation study was conducted to assess the finite-sample performance of the five estimators. The simulation procedure is as follows:

Step 1. Set sample sizes $n \in \{20, 30, 50, 100\}$ and parameter values $\theta \in \{1, 1.5, 2, 3\}$. The parameter values were chosen to represent distinct structural shapes of the Sujatha density. Recall that the polynomial factor $(x^2 + x + 1)$ controls the spread and shape of the density, while the exponential term governs the rate of tail decay. Consequently, $\theta = 1$ represents a moderate concentration with a noticeable right tail; $\theta = 1.5$ represents a transition regime between spread-out and concentrated shapes; $\theta = 2$ represents increased concentration with a shorter right tail; and $\theta = 3$ represents heavy concentration near zero with rapid exponential decay. These values therefore systematically explore the estimators' performance from heavier-tailed to more concentrated distributional shapes. (see Figure 1 and 2).

Step 2. Generate a random sample of size n from the Sujatha distribution with the specified θ using the inverse CDF method.

Step 3. Compute $\hat{\theta}$ via each of the five estimation procedures: MLE/MOM, MMOM, $PE_{0.25}$, $PE_{0.50}$, and $PE_{0.70}$.

Step 4. Repeat Steps 2–3 for N = 5,000 replications.

Step 5. Compute the Mean Square Error (MSE) for each estimator:

$$MSE\left(\hat{\theta}\right)=E\left[\left(\hat{\theta}-\theta\right)^2\right]=\frac{1}{N}\sum_k\left(\hat{\theta}_k-\theta\right)^2 \quad (11)$$

The MSE integrates both the squared bias and the variance of an estimator, making it a comprehensive measure of estimation accuracy. Lower MSE indicates higher efficiency.

3. Results and Discussion

Table 1 reports the MSE values for all estimators across the 16 simulation scenarios (4 sample sizes × 4 parameter values).

Table 1. Mean Square Error (MSE) of estimators for the Sujatha distribution parameter θ

Parameter (θ)	Sample Size (n)	MLE	MMOM	MOM	$PE_{0.25}$	$PE_{0.50}$	$PE_{0.75}$
1	20	0.05136	0.07059	0.06001	0.46712	0.12680	0.14612
	30	0.07005	0.44897	0.02158	1.05491	0.28599	0.33298
	50	0.61253	1.25269	0.16503	1.87355	0.51334	0.59012
	100	3.42589	4.27670	1.18532	4.22982	1.15194	1.33455
1.5	20	0.06618	0.09163	0.07678	0.61624	0.16501	0.18110
	30	0.06969	0.44896	0.02128	1.04462	0.28558	0.33337
	50	0.61403	1.25244	0.16437	1.87279	0.52191	0.58731
	100	3.42144	4.27695	1.18727	4.21379	1.15543	1.29229
2	20	0.05092	0.06739	0.05754	0.44776	0.11512	0.11650
	30	0.06871	0.44492	0.01515	1.08091	0.26751	0.25021
	50	0.61001	1.24746	0.15654	1.92674	0.47139	0.45668
	100	3.42090	4.26852	1.16528	4.30572	1.05269	1.03468
3	20	3.41795	4.27752	1.19003	4.21513	1.14352	1.22574
	30	3.41896	4.27042	1.17399	4.37437	1.05996	0.99765
	50	3.41909	4.26211	1.15048	4.40865	0.95607	0.78820
	100	3.41746	4.25829	1.14191	4.49262	0.90151	0.58174

MLE = Maximum Likelihood Estimator; MOM = Method of Moments; MMOM = Modified Method of Moments; PE = Percentile Estimator

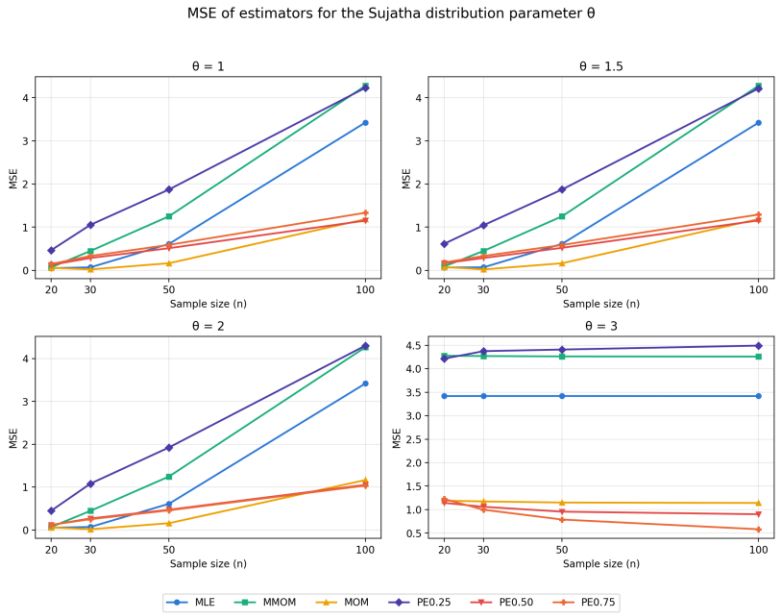


Figure 3: MSE vs. sample size, by θ

3.1 Performance of MLE and MOM

The MLE and MOM yield identical estimating equations for the Sujatha distribution, but produce different numerical estimates. For small samples (n = 20) and parameter values $\theta \leq 2$, MLE and MOM achieve the lowest MSE among all estimators, MLE with MSE value (0.05136) is more efficient than MOM with MSE value (0.06001). However, as the sample size increases (n > 20), the MOM gain the efficiency power over the MLE. The superior performance of MLE at very small samples (n = 20) is attributable to its use of the full likelihood function, which optimally extracts information from all observed data points simultaneously. However, MLE's advantage erodes as n increases because MOM, which directly matches the theoretical mean to the sample mean, becomes increasingly accurate as the sample mean converges to the population mean by the law of large numbers. The growing MSE of MLE with increasing n under fixed $\theta \leq 2$ reflects the numerical sensitivity of the Newton–Raphson solution to the cubic estimating equation, which can amplify small sample perturbations at larger n. These results are consistent with findings in comparable one-parameter lifetime distributions (Zaka & Akhter, 2014). Figure 3 illustrates these patterns.

3.2 Performance of MMOM

The MMOM estimator consistently exhibits moderate MSE values at $\theta \leq 2$ and $n \leq 50$. At large samples (n > 50) and parameter ($\theta = 3$), the MSE of MMOM is the highest in most scenarios, which suggest that it is not suitable for large parameter and large sample sizes. The deteriorating performance of MMOM at large θ and large n can be attributed to the sensitivity of the coefficient of variation (CV) estimator to extreme values. The CV is computed as the ratio of the sample standard deviation to the sample mean; at large θ, where observations are concentrated near zero, even small perturbations in the sample standard deviation produce large relative errors in the CV, which subsequently propagate into the fourth-degree polynomial estimating equation (9). This sensitivity is compounded at larger sample sizes where the variance of the sample CV is reduced but the systematic bias in matching the theoretical CV remains. These findings align with those of Zaka and Akhter (2014), who similarly observed that modified moment estimators based on higher-order distributional summaries can become less stable in heavy-concentration parameter regimes.

3.3 Performance of Percentile Estimators

Among the percentile estimators, the performance follows a clear hierarchy. $PE_{0.25}$ estimator consistently produces the highest value of MSE across all scenarios, making it the least efficient estimator overall. $PE_{0.50}$ and $PE_{0.75}$ perform comparably for small to moderate sample sizes. Notably, at $\theta = 3$ and n = 100, $PE_{0.50}$ achieves MSE = 0.90 and $PE_{0.75}$ achieves MSE = 0.58, both substantially lower than MOM (3.42) or MMOM (4.26). This makes the higher order percentile estimators competitive specifically for large θ and large n. The poor performance of $PE_{0.25}$ across all scenarios can be explained by the fact that the first quartile lies in the lower tail of the Sujatha density, where the CDF changes slowly and the estimating equation (10) is poorly conditioned, leading to high variance in the resulting estimates. In contrast, $PE_{0.75}$ and $PE_{0.50}$ lie in regions of steeper CDF slope, particularly for large θ where the distribution is concentrated near zero, making these quantiles more informative for estimating the parameter. The competitive advantage of higher-order percentile estimators for large θ is consistent with findings by Bhatti et al. (2018) for the Pareto distribution, where estimators based on order statistics in the upper portion of the distribution outperformed MLE and MOM in similar heavy-concentration regimes.

3.4 Overall Ranking

Based on the simulation results, the following general ranking of estimators emerges: MLE is good for small sample size (n=20); MOM is most efficient for $\theta \leq 2$ across all sample sizes; for $\theta = 3$, $PE_{0.75}$ and $PE_{0.50}$ with $n \geq 50$ provide the smallest MSE. The MMOM is a reliable middle ground choice for small-to-medium samples. $PE_{0.25}$ is consistently the least competitive method.

4. Conclusion

This paper presented a systematic comparison of six parameter estimation procedures for the one-parameter Sujatha distribution via a Monte Carlo simulation study. The main conclusions are as follows:

- MLE and MOM are equivalent for the Sujatha distribution and collectively represent the most efficient estimators when $\theta \leq 2$, regardless of sample size.
- For large parameter values ($\theta = 3$), the $PE_{0.50}$ and $PE_{0.75}$ estimators are markedly superior, particularly at larger sample sizes ($n \geq 50$).
- MMOM provides stable, moderate MSE performance and is recommended when sample sizes are small ($n \leq 30$) and $\theta \leq 2$.
- $PE_{0.25}$ is uniformly the least efficient method and is not recommended for practical use with this distribution.

Practitioners working with lifetime data modelled by the Sujatha distribution should therefore select their estimation method based on both the available sample size and prior knowledge about the likely magnitude of θ. Future work could explore Bayesian estimation, L-moment-based estimators, and the extension of these findings to censored data settings. It is important to note several scope limitations of this study. First, the analysis is restricted to univariate, complete (uncensored) data; the performance of these estimators under Type I or Type II censoring schemes, which are common in reliability and survival analysis applications, remains unknown. Secondly, only finite-sample MSE is considered; asymptotic efficiency comparisons and confidence interval coverage are not assessed. Third, the study is confined to the original one-parameter Sujatha distribution and does not address generalized or multi-parameter extensions.
Future research should address these gaps by:

(i) incorporating Bayesian estimators with informative and non-informative prior distributions, specifying, for example, a Gamma prior on θ, to assess whether prior information substantially improves estimation under small samples;
(ii) Investigating L-moment-based estimators, which are known to be robust to outliers and may perform well for the heavier-tailed configurations of the Sujatha distribution;
(iii) Evaluating estimator performance under censored data scenarios using the maximum product of spacings or expectation–maximization approaches; and
(iv) Applying the recommended estimation procedure to real-world lifetime datasets to validate the simulation findings and demonstrate practical utility.

This is the first comprehensive Monte Carlo simulation study that systematically compares six parameter estimation procedures namely; MLE, MOM, MMOM, $PE_{0.25}$, $PE_{0.50}$, and $PE_{0.75}$ for the one-parameter

Sujatha distribution originally proposed by Shanker (2016). Prior work on this distribution considered only MLE and MOM (Shanker, 2016) or focused on multi-parameter extensions (Nwikpe & Iwok, 2020), leaving a significant gap in understanding the finite-sample efficiency of alternative estimators. This study fills that gap by providing clear, simulation-validated practical guidelines for selecting the most efficient estimator depending on sample size and parameter regime, thereby advancing the methodological toolkit available to practitioners in lifetime data analysis and reliability engineering.

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