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Fixed Point Analysis in Banach Spaces for Generalized Mappings

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Abstract

This paper establishes the existence and uniqueness of fixed points for a new class of mappings in Banach spaces, involving fixed point analysis for generalized mappings. These mappings are inspired by the interplay of classical contraction principles and advanced concepts in functional analysis. We introduce novel conditions that ensure fixed point properties in the context of Banach spaces. The findings have potential applications in areas such as matrix algebra, combinatorial set theory, and building technology, where tower-like structures arise naturally. The results extend, improve, generalize, and unify various existing fixed point theorems in literature.

Keywords: Unique fixed point, Banach spaces, existence, generalize mappings.

1 Introduction

The literature on the concept of tower functions is very scanty but special in Mathematics, spanning areas such as real analysis, matrix algebra, and combinatorial set theory. For instance, functions of the form $n \mapsto n^n$ or matrices of the type A^A are classical examples. These functions, though seemingly elementary, present significant challenges in evaluation and understanding. Similarly, in metric and normed spaces, mappings exhibiting such behavior have garnered recent attention.

Fixed point theory is a fundamental topological and analytical tool of immense importance in mathematics, providing the theoretical backbone for proving the existence and uniqueness of solutions to differential equations, integral equations, and optimization problems. It owes its origins to the Banach Contraction Principle (BCP), established in 1922 by Stefan Banach. BCP asserts that a contraction mapping $f : X \to X$ on a complete metric space X has a unique fixed point, provided the contraction condition $d(f(x), f(y)) \leq Ld(x, y)$, where $0 \leq L < 1$, holds.

While these classical contraction principles are powerful, they possess significant limitations. Primarily, the BCP requires a strict, uniform contraction constant less than one. This restrictive condition renders the classical theory inadequate for maps that are only locally contractive, nonexpansive, or that exhibit variable asymptotic contraction behavior across complex spaces.

To circumvent these limitations, we introduce a new class of generalized tower mappings. The motivation behind this study is to relax classical constraints by utilizing auxiliary control functions and nested iteration powers. This enables a much richer mathematical framework capable of handling complex non-linear operators that emerge in advanced applied mathematics. This principle Banach (1922); Cho, Bae, & Karapinar (2013); Popescu (2014); Geraghty (1973) has since been generalized extensively. Among these, Cho, Bae, & Karapinar (2013) introduced α-Geraghty contractions, while Popescu (2014) expanded on these results using alternative conditions.

Recently, Okeke & Francis (2023); Francis & Okeke (2024) advanced the study of tower mappings in metric spaces, proving fixed point results for Geraghty tower-type mappings. These contributions build on foundational work by Geraghty (1973) and others Banach (1922); Cho, Bae, & Karapinar (2013). Extending these works, this paper investigates the existence and uniqueness of fixed points for mappings involving finite and infinite powers in Banach spaces. The results herein aim to provide a comprehensive framework that bridges classical contraction principles with new directions in fixed point theory.

In fact, there are large amount of literature dealing with extensions or generalizations of this remarkable theorem. This has encountered in so many extensions/generalizations to mention a few as recorded in Amini-Harandi & Emami (2010); Chu & Diaz (1965); Kannan (1968); Reich (1972, 1971); Geraghty (1973); Dass & Gupta (1975); Jaggi (1977); Meir & Keeler (1969) and their references therein. Cho, Bae, & Karapinar (2013) defined the concept of α-Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Popescu (2014) generalized the results obtained in Cho, Bae, & Karapinar (2013) and gave other conditions to prove the existence and uniqueness of a fixed point of α-Geraghty contraction type maps in the context of a complete metric space. See also Okeke, Francis, & Gibali (1922); Okeke, Francis, de la Sen, & Abbas (2021); Okeke & Francis (2021); Okeke, Francis, & de la Sen (2020) for other results in fixed point theory.

Okeke & Francis (2023) defined metric tower mappings in metric spaces and proved the existence of fixed point of Geraghty tower-type mappings with example in complete metric spaces. In the same Sprite, Francis & Okeke (2024) defined rational type Geraghty tower contraction mapping and proved the existence of such rational Geraghty tower results in the context of complete metric spaces.

The paper is structured as follows: Section 2 outlines the preliminary definitions and foundational results necessary for the subsequent analysis. Section 3 presents the main results, with detailed proofs.

2 Preliminaries

Throughout the article $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\}$ is the set of non-negative integers and $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ is the set of positive real numbers.

To build a contextual framework for our new mappings, it is essential to highlight relevant

existing fixed point theorems. Following Banach's work, Kannan demonstrated that a mapping can have a unique fixed point without being continuous. Furthermore, Geraghty replaced the standard contraction constant with a class of auxiliary functions mapping to $[0,1)$, ensuring convergence without a strictly fixed upper bound. Our generalized mappings integrate these rich historical concepts specifically the structural flexibility of Banach spaces and the non-linear bounding of Geraghty functions to analyze tower-type functional iterations.

Definition 1. *Pathak (2018) Let X be a linear space over field $\mathbb{K}$ ($\mathbb{K}$ means $\mathbb{R}$ or $\mathbb{C}$) and $\|\cdot\| : X \to \mathbb{R}^+$ a function. Then, $\|\cdot\|$ is said to be a norm defined on X if the following properties hold:*
$(N1)$ $\|x\| = 0$ *if and only if* $x = 0$*; (strict positivity)*
$(N2)$ $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in X$ *and* $\lambda \in \mathbb{K}$*; (absolute homogeneity)*
$(N3)$ $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$*. (triangle inequality or subadditivity).*

The ordered pair $(X,\ \|\cdot\|)$ is *called* a *normed linear space (abbreviated nls) or simply a normed space.*

Remark 1. (1) $\|x\| \geq 0$ and $\|-x\| = \|x\|$ for all x in X.
(2) $\big|\|x\| - \|y\|\big| \leq \|x-y\|$ and $\big|\|x\| - \|y\|\big| \leq \|x+y\|$ for all x, y in X.
(3) $\|\cdot\|$ is a continuous function; i.e., $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} \|x_n\| = \|x\|$.
(4) Norm is a convex function; i.e., $\|\alpha x + (1-\alpha)y\| \leq \alpha\|x\| + (1-\alpha)\|y\|$ for all $x, y \in X$ and $\alpha \in [0,1]$.
(5) Every normed space $(X,\ \|\cdot\|)$ is a metric space with induced metric d defined by $d(x,\ y) = \|x-y\|$ for all x, y in X.
The induced metric in turn defines a topology on X called the norm topology.
(6) If a linear space is equipped with the metric function $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases}$$

it follows that the linear space is a metric space.

Definition 2. *Pathak (2018) A sequence $\{x_n\}$ in a normed space X is said to be convergent to $x \in X$ if $\lim_{n\to\infty} \|x_n - x\| = 0$, i.e., given an $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that*

$$\|x_n - x\| < \epsilon,\ \forall n \geq n_0.$$

To show that $\{x_n\}$ converges to x, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 3. *Pathak (2018) (Cauchy sequence) A sequence $\{x_n\}$ in a normed space X is a Cauchy sequence* if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$, i.e., *given an $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that*

$$\|x_n - x_m\| < \epsilon,\ \forall n,\ m \geq n_0.$$

Theorem 2. *Pathak (2018) Every convergent sequence in a normed space is Cauchy but not every Cauchy sequence converges*

Proof. Let $\{x_n\}$ be a sequence in a normed space X and let $x_n \to x$. Then, for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$\|x_n - x\| < \frac{\epsilon}{2}$ for all $n \geq n_0$.

Hence, for $m, n \geq n_0$, we have

$$\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that every convergent sequence is Cauchy in a normed space X. For the converse, see Pathak (2018). □

Remark 3. Normed spaces where every Cauchy sequence is guaranteed to converge are called Banach spaces. A classical example of a normed space that is not Banach, i.e. Cauchy sequence fail to converge is the space of polynomials on the interval $[0,1]$ equipped with the supremum norm, as the limit of certain Cauchy sequences of polynomials requires an infinite degree polynomial like e^x which leaves the space.

Definition 4. *Pathak (2018) A normed space is said to be complete if every Cauchy sequence in it is convergent, i.e., for every Cauchy sequence $\{x_n\}$ in X, there is an element x in X such that $x_n \to x$.*

Definition 5. *Geraghty (1973) S is the class of functions $\alpha : \mathbb{R}_+ \to [0,1)$ with*
(i) $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t > 0\}$,
(ii) $\alpha(t_n) \to 1 \implies t_n \to 0$ as $n \to \infty$.

Definition 6. *Chaipunya, Cho, & Kumam (2012) For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the class of n-tuples of functions $(\beta_1, \beta_2, \beta_3, \cdots, \beta_n)$, where for each $i \in \{1, 2, \cdots, n\}$, $\beta_i : \mathbb{R}_+ \cup \{\infty\} \to [0,1)$ and the following implications holds: $\beta(t_k) := \beta_1(t_k) + \beta_2(t_k) + \cdots + \beta_n(t_k) \to 1$ implies $t_k \to 0$.*

Definition 7. *Okeke & Francis (2023) Let (X,d) be a metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$. Then T is called a metric tower map if*

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}; \tag{1}$$

where, $A_\beta := \beta(d(x,y))d(x,Tx)$, $B_\gamma := \gamma(d(x,y))d(y,Ty)$, $C_\delta := \delta(d(x,y))d(x,Ty)$; $D_\epsilon := \epsilon(d(x,y))d(y,Tx)$, for all distinct $x, y \in X$

Definition 8. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$. Then T is called a normed tower map if*

$$\|Tx - Ty\| \leq \alpha(\|x - y\|)\|x - y\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}; \tag{2}$$

where, $A_\beta := \beta(\|x - y\|)\|x - Tx\|$, $B_\gamma := \gamma(\|x - y\|)\|y - Ty\|$, $C_\delta := \delta(\|x - y\|)\|x - Ty\|$;
$D_\epsilon := \epsilon(\|x - y\|)\|y - Tx\|$, for all distinct $x, y \in X$.

3 Main Results

Theorem 4. *Let $(X,\|\cdot\|)$ be a Banach space and $T:X\to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\}\in\mathcal{F}_{Ger}$ satisfying;*

$$\|Tx-Ty\|^2\leq\alpha(\|x-Ty\|)\|x-y\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}; \tag{3}$$

where, $A_\beta:=\dfrac{\beta(\|x-Ty\|)\|x-Tx\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, $B_\gamma:=\dfrac{\gamma(\|x-Ty\|)\|y-Ty\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, $C_\delta:=\dfrac{\delta(\|x-Ty\|)\|x-Ty\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$; $D_\epsilon:=\dfrac{\epsilon(\|x-Ty\|)\|y-Tx\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, *for* $x,y\in X$ *and* $\ln(\|x-y\|)\leq\|x-y\|$, *so that there exists* $\theta>0$, $\|x\tilde{-}y\|=\frac{\|x-y\|}{\theta}$ *for which* $\theta\geq\sup_{x,y\in X}\|x-y\|$, $\|x\tilde{-}y\|^k\leq\|x\tilde{-}y\|$ *for all* $k\in\mathbb{N}$. *Then the sequence* $\{x_k\}_{k\in\mathbb{N}}$ *defined by* $Tx_{k-1}=x_k$ *converges to a unique fixed point of* T *in* X.

Proof. Suppose, if possible, otherwise. Let $x_k=Tx_{k-1}$, then $x_{k+1}=Tx_k$ for all $k\in\mathbb{N}$. But $x_k\neq x_{k+1}$ and this implies that $\|x_k-x_{k+1}\|>0$, so that $\|x_k-x_{k+1}\|=\|Tx_{k-1}-Tx_k\|$. From inequality (3), take $x=x_{k-1}$ and $y=x_k$ for all $k\in\mathbb{N}$, thus

$$\begin{aligned}\|x_k-x_{k+1}\|^2&=\|Tx_{k-1}-Tx_k\|^2\\&\leq\alpha(\|x_{k-1}-Tx_k\|)\|x_{k-1}-x_k\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}};\end{aligned} \tag{4}$$

where;

$A_\beta:=\dfrac{\beta(\|x_{k-1}-Tx_k\|)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$, $B_\gamma:=\dfrac{\gamma(\|x_{k-1}-Tx_k\|)\|x_k-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$,
$C_\delta:=\dfrac{\delta(\|x_{k-1}-Tx_k\|)\|x_{k-1}-Tx_k\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$; $D_\epsilon:=\dfrac{\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$.
So that, $A_\beta:=\dfrac{\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$, $B_\gamma:=\dfrac{\gamma(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$,
$C_\delta:=\dfrac{\delta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_{k+1}\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$; $D_\epsilon:=\dfrac{\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$.
Thus, $A_\beta:=\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|$, $B_\gamma:=\gamma(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|$,
$C_\delta:=\delta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_{k+1}\|$; $D_\epsilon:=\dfrac{\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$.
Therefore we have by taking $W_{\|\Delta x_k\|}:=\|x_k-x_{k+1}\|^2$, thus inequality (4) reduced to

$$\begin{aligned}W_{\|\Delta x_k\|}&\leq\alpha(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|^{A_\beta^{B_\gamma}}\\&\leq\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\left(A_\beta^{B_\gamma}\ln(\|x_k-x_{k-1}\|)\right)^j}{j!}\\&=\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{A_\beta^{jB_\gamma}\ln^j(\|x_k-x_{k-1}\|)}{j!}\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{A_{\beta}^{jB_\gamma}\|x_k-x_{k-1}\|}{j!}\\
&= \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\exp\left(jB_\gamma\ln(A_\beta)\right)\|x_k-x_{k-1}\|}{j!}\\
&= \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\|x_k-x_{k-1}\|}{j!}\sum_{r=0}^{\infty}\frac{\left(jB_\gamma\ln(A_\beta)\right)^r}{r!}\\
&= \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\|x_k-x_{k-1}\|}{j!}\sum_{r=0}^{\infty}\frac{j^rB_\gamma\ln^r(A_\beta)}{r!}\\
&= \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\|x_k-x_{k-1}\|}{j!}\sum_{r=0}^{\infty}\frac{j^r\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|^r\times \ln^r(\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|)}{r!}\\
&= \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|^r\times \ln^r(\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|)\|x_k-x_{k-1}\|}{j!r!}\\
&\leq \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|^r\\
&\quad \ln^r(\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|)\|x_k-x_{k-1}\|\\
&\leq \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|\\
&\quad \beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|\|x_k-x_{k-1}\|\\
&\leq \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|\beta(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k-1}\|\\
&= \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|\|x_k-x_{k-1}\|.
\end{aligned}
$$

Hence;

$$
\begin{aligned}
\|x_k-x_{k+1}\| &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}j^r\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k-1}\|\\
&= \left(\sum_{j=0}^{\infty}\frac{j^r\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)}{1-\gamma^j(\|x_{k-1}-x_{k+1}\|)}\right)\|x_k-x_{k-1}\|. \qquad (5)
\end{aligned}
$$

Therefore, from inequality (5), we get;

$$
\|x_k-x_{k+1}\| \leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k-1}\|. \qquad (6)
$$

Therefore,

$$\begin{aligned}\|x_k - x_{k+1}\| &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k - x_{k-1}\| \\ &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-2}-x_k\|)\beta(\|x_{k-2}-x_k\|)\gamma^r(\|x_{k-2}-x_k\|)\|x_{k-1} - x_{k-2}\| \\ &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-3}-x_{k-1}\|)\beta(\|x_{k-3}-x_{k-1}\|)\gamma^r(\|x_{k-3}-x_{k-1}\|)\|x_{k-2} - x_{k-3}\| \\ &\vdots \\ &\leq \|x_0 - x_1\|. \end{aligned} \tag{7}$$

Thus, $\{\|x_k - x_{k+1}\|\}_{k\in\mathbb{N}}$ is a non-increasing sequence and bounded below for which the sequence converges to some real number $a \geq 0$ (say). If $a > 0$, then we see that

$$\|x_k - x_{k+1}\| \leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k - x_{k-1}\|. \tag{8}$$

Thus,

$$1 \leq \lim_{k\to\infty}\inf\left\{\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\right\}. \tag{9}$$

This implies that $\lim_{k\to\infty}\|x_{k-1} - x_{k+1}\| = 0$, which is a contradiction. Therefore,

$$\lim_{k\to\infty}\|x_k - x_{k+1}\| = 0. \tag{10}$$

Now, we shall show that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Suppose that there exists an $\epsilon^* > 0$ for which we define two sub-sequences $\{x_{k_j}\}_{j\in\mathbb{N}}$ and $\{x_{k_h}\}_{h\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that for $k_j > k_h > k$, $x_{k_j} \geq x_{k_j-1}$ and $x_{k_j} \neq x_{k_h}$ so that $\left\|x_{k_j} - x_{k_h}\right\| > \epsilon^*$ and $\left\|x_{k_{j-1}} - x_{k_j}\right\| < \eta$, $\left\|x_{k_{h-1}} - x_{k_h}\right\| < \vartheta$. Now $\left\|x_{k_j} - x_{k_h}\right\|^2 = \left\|Tx_{k_{j-1}} - Tx_{k_{h-1}}\right\|^2$. Take $x = x_{k_{j-1}}$ and $y = x_{k_{h-1}}$, so inequality (3) becomes;

$$\begin{aligned}\left\|x_{k_j} - x_{k_h}\right\|^2 &= \left\|Tx_{k_{j-1}} - Tx_{k_h}\right\|^2 \\ &\leq \alpha\Big(\left\|x_{k_{j-1}} - x_{k_{h-1}}\right\|\Big)\left\|x_{k_{j-1}} - x_{k_{h-1}}\right\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}; \end{aligned} \tag{11}$$

where,

$$A_\beta := \frac{\beta\Big(\left\|x_{k_{j-1}} - x_{k_h}\right\|\Big)\left\|x_{k_{j-1}} - Tx_{k_{j-1}}\right\|}{1+\epsilon\Big(\left\|x_{k_{j-1}} - x_{k_h}\right\|\Big)\left\|x_{k_{h-1}} - Tx_{k_{j-1}}\right\|},\ B_\gamma := \frac{\gamma\Big(\left\|x_{k_{j-1}} - x_{k_h}\right\|\Big)\left\|x_{k_{h-1}} - Tx_{k_{h-1}}\right\|}{1+\epsilon\Big(\left\|x_{k_{j-1}} - x_{k_h}\right\|\Big)\left\|x_{k_{h-1}} - Tx_{k_{j-1}}\right\|},$$

$$C_\delta := \frac{\delta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - Tx_{k_{h-1}}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - Tx_{k_{j-1}}\right\|};\ D_\epsilon := \frac{\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - Tx_{k_{j-1}}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - Tx_{k_{j-1}}\right\|}.$$

Hence,

$$A_\beta := \frac{\beta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_j}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|},\ B_\gamma := \frac{\gamma\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_h}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|},$$

$$C_\delta := \frac{\delta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_h}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|};\ D_\epsilon := \frac{\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|},$$

So that,

$$A_\beta := \frac{\beta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_j}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|} \leq \beta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_j}\right\|,$$

$$B_\gamma := \frac{\gamma\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_h}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|} \leq \gamma\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_h}\right\|,$$

$$C_\delta := \frac{\delta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_h}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|} \leq \delta\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_h}\right\|;$$

$$D_\epsilon := \frac{\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|}{1+\epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|} \leq \epsilon\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{h-1}} - x_{k_j}\right\|.$$

Thus, from inequality (11), we get by letting

$$t_{k_j} = \left\|x_{k_j} - x_{k_h}\right\|^2$$

$$\begin{aligned}
t_{k_j} &= \left\|Tx_{k_{j-1}} - Tx_{k_{h-1}}\right\|^2 \\
&\leq \alpha\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\left\|x_{k_{j-1}} - x_{k_{h-1}}\right\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}} \\
&= \alpha\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\exp\left\{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}\ \ln\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\right\} \\
&= \alpha\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\sum_{i=0}^{\infty}\frac{\left\{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}\ \ln\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\right\}^i}{i!} \\
&= \alpha\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\sum_{i=0}^{\infty}\frac{A_\beta^{iB_\gamma^{C_\delta^{D_\epsilon}}}\ \ln^i\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)}{i!} \\
&= \alpha\left(\left\|x_{k_{j-1}} - x_{k_h}\right\|\right)\sum_{i=0}^{\infty}\frac{\ln^i\left(\left\|x_{k_{j-1}} - x_{k_{h-1}}\right\|\right)}{i!}\exp\left(iB_\gamma^{C_\delta^{D_\epsilon}}\ \ln(A_\beta)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \frac{\ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big)}{i!} \sum_{r=0}^{\infty} \frac{\left(iB_\gamma^{C_\delta^{D_\epsilon}} \ln(A_\beta)\right)^r}{r!} \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \frac{\ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big)}{i!} \sum_{r=0}^{\infty} \frac{i^r \ln^r(A_\beta)}{r!} B_\gamma^{rC_\delta^{D_\epsilon}} \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \frac{\ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big)}{i!} \sum_{r=0}^{\infty} \frac{i^r \ln^r(A_\beta)}{r!} \exp\left(rC_\delta^{D_\epsilon} \ln(B_\gamma)\right) \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \frac{\ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big)}{i!} \sum_{r=0}^{\infty} \frac{i^r \ln^r(A_\beta)}{r!} \sum_{m=0}^{\infty} \frac{r^m C_\delta^{mD_\epsilon} \ln^m(B_\gamma)}{m!} \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \frac{\ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big)}{i!} \sum_{r=0}^{\infty} \frac{i^r \ln^r(A_\beta)}{r!} \sum_{m=0}^{\infty} \frac{r^m \ln^m(B_\gamma)}{m!} \exp\Big(mD_\epsilon \ln(C_\delta)\Big) \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \frac{\ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big)}{i!} \sum_{r=0}^{\infty} \frac{i^r \ln^r(A_\beta)}{r!} \sum_{m=0}^{\infty} \frac{r^m \ln^m(B_\gamma)}{m!} \sum_{n=0}^{\infty} \frac{m^n D_\epsilon^n \ln^n(C_\delta)}{n!} \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n \ln^r(A_\beta) \ln^m(B_\gamma) \ln^n(C_\delta)}{i!r!m!n!} D_\epsilon^n \ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big) \\
&\leq \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n}{i!r!m!n!} \ln^r(A_\beta) \ln^m(B_\gamma) \ln^n(C_\delta) D_\epsilon^n \ln^i\Big(\Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|\Big) \\
&\leq \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n}{i!r!m!n!} A_\beta^r B_\gamma^m C_\delta^n D_\epsilon^n \Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\| \\
&\leq \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n}{i!r!m!n!} \beta^r\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{j-1}} - x_{k_j}\Big\| \gamma^m\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \\
&\times \Big\|x_{k_{h-1}} - x_{k_h}\Big\| \delta^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{j-1}} - x_{k_h}\Big\| \Big(\epsilon\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{h-1}} - x_{k_j}\Big\|\Big)^n \\
&\times \Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\| \\
&= \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n}{i!r!m!n!} \beta^r\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{j-1}} - x_{k_j}\Big\| \gamma^m\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \\
&\times \Big\|x_{k_{h-1}} - x_{k_h}\Big\| \delta^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{j-1}} - x_{k_h}\Big\|^n \epsilon^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{h-1}} - x_{k_j}\Big\| \\
&\times \Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\| \\
&\leq \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n}{i!r!m!n!} \beta^r\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{j-1}} - x_{k_j}\Big\| \gamma^m\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \\
&\times \Big\|x_{k_{h-1}} - x_{k_h}\Big\| \delta^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{j-1}} - x_{k_h}\Big\| \epsilon^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{h-1}} - x_{k_j}\Big\| \\
&\times \Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\| \\
&\leq \eta\vartheta \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^r r^m m^n}{i!r!m!n!} \alpha\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \beta^r\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \gamma^m\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \\
&\delta^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \epsilon^n\Big(\Big\|x_{k_{j-1}} - x_{k_h}\Big\|\Big) \Big\|x_{k_{h-1}} - x_{k_j}\Big\| \Big\|x_{k_{j-1}} - x_{k_h}\Big\| \Big\|x_{k_{j-1}} - x_{k_{h-1}}\Big\|
\end{aligned}
$$

$$\leq \eta\vartheta \sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^r r^m m^n}{i!r!m!n!}\alpha\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\beta^r\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\gamma^m\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)$$
$$\delta^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\epsilon^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big\|x_{k_{j-1}}-x_{k_{h-1}}\Big\|$$
$$\leq \eta\vartheta \sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^r r^m m^n}{i!r!m!n!}\alpha\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\beta^r\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\gamma^m\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)$$
$$\delta^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\epsilon^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\Big\|x_{k_j}-x_{k_h}\Big\|\Big\|x_{k_j}-x_{k_{h-1}}\Big\|. \tag{12}$$

From inequality (12), we have

$$\Big\|x_{k_j}-x_{k_h}\Big\|^2 \leq \eta\vartheta \sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^r r^m m^n}{i!r!m!n!}\alpha\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\beta^r\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\gamma^m\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)$$
$$\delta^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\epsilon^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\Big\|x_{k_j}-x_{k_{h-1}}\Big\|. \tag{13}$$

From inequality (13), we get;

$$\frac{1}{\eta\vartheta} \leq \lim_{j,h\to\infty}\inf\Bigg\{\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^r r^m m^n}{i!r!m!n!}\alpha\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\beta^r\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\gamma^m\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)$$
$$\delta^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\epsilon^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\Bigg\}, \tag{14}$$

so that

$$\frac{1}{\sigma} \leq \lim_{j,h\to\infty}\inf\Bigg\{\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^r r^m m^n}{i!r!m!n!}\alpha\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\beta^r\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\gamma^m\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)$$
$$\delta^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\epsilon^n\Big(\Big\|x_{k_{j-1}}-x_{k_h}\Big\|\Big)\Bigg\}, \tag{15}$$

as $\sigma = \eta\vartheta$, which implies that $\lim_{h,j\to\infty}\Big\|x_{k_{j-1}}-x_{k_h}\Big\| = 0$, so that

$$\lim_{h,j\to\infty}\Big\|x_{k_j}-x_{k_h}\Big\| < \epsilon^*. \tag{16}$$

This is a contradiction, hence $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in Banach space X. Since Banach space X is complete, there is a candidate $x^* \in X$ such that $x_k \to x^* \in X$. Now we show that x^* is a fixed point of T. Suppose that $Tx^* \neq x^*$ and $\|x_k - Tx^*\| = \|Tx_{k-1} - Tx^*\|$. From inequality (3), take $x = x_{k-1}$ and $y = x^*$ for all $k \in \mathbb{N}$, thus;

$$\|x_k - Tx^*\|^2 = \|Tx_{k-1} - Tx^*\|^2$$
$$\leq \alpha\Big(\|x_{k-1} - Tx^*\|\Big)\|x_{k-1} - x^*\|^{A_{\beta}^{B_{\gamma}^{C_{\delta}^{D_{\epsilon}}}}}; \tag{17}$$

where;

$$A_\beta := \frac{\beta\Big(\|x_{k-1}-Tx^*\|\Big)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx_{k-1}\|},\ B_\gamma := \frac{\gamma\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx^*\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx_{k-1}\|},$$

$$C_\delta := \frac{\delta\Big(\|x_{k-1}-Tx^*\|\Big)\|x_{k-1}-Tx^*\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx_{k-1}\|};\ D_\epsilon := \frac{\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx_{k-1}\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx_{k-1}\|}.$$

Thus, $A_\beta := \dfrac{\beta\Big(\|x_{k-1}-Tx^*\|\Big)\|x_{k-1}-x_k\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-x_k\|} \leq \beta\Big(\|x_{k-1}-Tx^*\|\Big)\|x_{k-1}-x_k\|,$

$$B_\gamma := \frac{\gamma\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx^*\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-x_k\|} \leq \gamma\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-Tx^*\|,$$

$$C_\delta := \frac{\delta\Big(\|x_{k-1}-Tx^*\|\Big)\|x_{k-1}-Tx^*\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-x_k\|} \leq \delta\Big(\|x_{k-1}-Tx^*\|\Big)\|x_{k-1}-Tx^*\|;$$

$$D_\epsilon := \frac{\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-x_k\|}{1+\epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-x_k\|} \leq \epsilon\Big(\|x_{k-1}-Tx^*\|\Big)\|x^*-x_k\|.$$

From inequality (16), we get;

$$\|x^*-Tx^*\|^2 = 0, \tag{18}$$

this is a contradiction. Therefore, $x^* = Tx^*$. For the uniqueness, suppose that the fixed point of T in X that is $x^* = Tx^*$ for some $x^* \in Fix(T)$. Again, suppose if possible that there exists another fixed point of T in X, that is $y^* = Ty^*$ for some $y^* \in Fix(T)$. Now suppose that $\{x^* \neq y^*\} \Longrightarrow \|x^*-y^*\| > 0$. Now, we know that

$$\begin{aligned}\|x^*-y^*\|^2 &= \lim_{k\to\infty}\|x_k-y^*\|^2\\ &= \lim_{k\to\infty}\|Tx_{k-1}-Ty^*\|^2\\ &\leq \lim_{k\to\infty}\alpha\Big(\|x_{k-1}-Ty^*\|\Big)\|x_{k-1}-y^*\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}\end{aligned} \tag{19}$$

where,

$$A_\beta := \frac{\beta\Big(\|x_{k-1}-Ty^*\|\Big)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon\Big(\|x_{k-1}-Ty^*\|\Big)\|y^*-Tx_{k-1}\|},\ B_\gamma := \frac{\gamma\Big(\|x_{k-1}-Ty^*\|\Big)\|y^*-Ty^*\|}{1+\epsilon\Big(\|x_{k-1}-Ty^*\|\Big)\|y^*-Tx_{k-1}\|},$$

$$C_\delta := \frac{\delta\Big(\|x_{k-1}-Ty^*\|\Big)\|x_{k-1}-Ty^*\|}{1+\epsilon\Big(\|x_{k-1}-Ty^*\|\Big)\|y^*-Tx_{k-1}\|};\ D_\epsilon := \frac{\epsilon\Big(\|x_{k-1}-Ty^*\|\Big)\|y^*-Tx_{k-1}\|}{1+\epsilon\Big(\|x_{k-1}-Ty^*\|\Big)\|y^*-Tx_{k-1}\|},$$

so that $A_\beta := \dfrac{\beta\Big(\|x_{k-1}-y^*\|\Big)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon\Big(\|x_{k-1}-y^*\|\Big)\|y^*-Tx_{k-1}\|} \leq \beta\Big(\|x^*-y^*\|\Big)\|x_{k-1}-x_k\|,$

$$B_\gamma := \frac{\gamma\Big(\|x_{k-1}-y^*\|\Big)\|y^*-y^*\|}{1+\epsilon\Big(\|x_{k-1}-y^*\|\Big)\|y^*-Tx_{k-1}\|} \leq \gamma\Big(\|x_{k-1}-y^*\|\Big)\|y^*-y^*\| = 0,$$

$$C_\delta := \frac{\delta\Big(\|x_{k-1}-y^*\|\Big)\|x_{k-1}-y^*\|}{1+\epsilon\Big(\|x_{k-1}-y^*\|\Big)\|y^*-Tx_{k-1}\|} \leq \delta\Big(\|x_{k-1}-y^*\|\Big)\|x_{k-1}-y^*\|;$$
$$D_\epsilon := \frac{\epsilon\Big(\|x_{k-1}-y^*\|\Big)\|y^*-Tx_{k-1}\|}{1+\epsilon\Big(\|x_{k-1}-y^*\|\Big)\|y^*-Tx_{k-1}\|} \leq \epsilon\Big(\|x_{k-1}-y^*\|\Big)\|y^*-x_k\|.$$

We can see clearly from inequality (19) as $k \to \infty$, collapsed to

$$\|x^*-y^*\|^2 \leq \alpha\Big(\|x^*-y^*\|\Big)\|x^*-y^*\|, \tag{20}$$

Thus;

$$\|x^*-y^*\| \leq \alpha(\|x^*-y^*\|)\|x^*-y^*\|. \tag{21}$$

So, $\alpha\Big(\|x^*-y^*\|\Big) \to 1 \implies \|x^*-y^*\| = 0$. Therefore,

$$\|x^*-y^*\| = 0, \tag{22}$$

which is a contradiction. Hence equation (22) shows that the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X and the proof is completed. □

Remark 5. The power of the generalized inequality (3) lies in its ability to reduce to foundational fixed point theorems by placing specific restrictions on the Geraghty function classes ($\mathcal{F}_{Ger}$) and the exponent tower.

(1) Banach Contraction Condition: Suppose we restrict our functions such that $\alpha(\cdot) = L^2$ where $L \in [0,1)$ (which is a valid constant function in the Geraghty class). Furthermore, assume the auxiliary functions $(\beta,\gamma,\delta,\epsilon)$ are configured such that the entire rational exponent tower evaluates to 2, i.e., $A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}} = 2$. Under these conditions, the main inequality (3) reduces to

$$\|Tx-Ty\|^2 \leq L^2\|x-y\|^2. \tag{23}$$

Taking the square root of both sides immediately yields:

$$\|Tx-Ty\| \leq L\|x-y\|, \tag{24}$$

which makes the classical Banach contraction condition in Banach spaces evident.

(2) Geraghty Contraction Condition: If we allow $\alpha \in \mathcal{F}_{Ger}$ to remain a standard Geraghty function (rather than a constant), but again restrict the tower components such that $A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}} = 2$, substituting this into (3) yields

$$\|Tx-Ty\|^2 \leq \alpha(\|x-Ty\|)\|x-y\|^2. \tag{25}$$

Taking the square root gives:

$$\|Tx-Ty\| \leq \sqrt{\alpha(\|x-Ty\|)}\|x-y\|. \tag{26}$$

Because the square root of a Geraghty function mapping to $[0,1)$ maintains the fundamental properties of the Geraghty class, this isolates the classical Geraghty contraction condition within the context of a Banach space.

Corollary 6. *Let $(X,\|\cdot\|)$ be a Banach space and $T:X\to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\}\in\mathcal{F}_{Ger}$ satisfying;*

$$\|Tx-Ty\|^2\leq\alpha(\|x-Ty\|)\|x-y\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}; \tag{27}$$

where, $A_\beta:=\dfrac{\beta(\|x-Ty\|)\|x-Tx\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, $B_\gamma:=\dfrac{\gamma(\|x-Ty\|)\|y-Ty\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$,
$C_\delta:=\dfrac{\delta(\|x-Ty\|)\|x-Ty\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$; $D_\epsilon:=\dfrac{\epsilon(\|x-Ty\|)\|y-Tx\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, *for all distinct $x,y\in X$ and* $\ln(\|x-y\|)\leq\|x-y\|$, *so that there exists $\theta>0$,* $\|x\tilde{-}y\|=\frac{\|x-y\|}{\theta}$ *for which* $\theta\geq\sup_{x,y\in X}\|x-y\|$, $\|x\tilde{-}y\|^k\leq\|x\tilde{-}y\|$ *for all $k\in\mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1}=x_k$ converges to a unique fixed point of T in X.*

Proof. Suppose, if possible, otherwise. Let $x_k=Tx_{k-1}$, then $x_{k+1}=Tx_k$ for all $k\in\mathbb{N}$. But $x_k\neq x_{k+1}$ and this implies that $\|x_k-x_{k+1}\|>0$, so that $\|x_k-x_{k+1}\|=\|Tx_{k-1}-Tx_k\|$. From inequality (27), take $x=x_{k-1}$ and $y=x_k$ for all $k\in\mathbb{N}$, thus

$$\begin{aligned}\|x_k-x_{k+1}\|^2&=\|Tx_{k-1}-Tx_k\|^2\\&\leq\alpha(\|x_{k-1}-Tx_k\|)\|x_{k-1}-x_k\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}};\end{aligned} \tag{28}$$

where;
$A_\beta:=\dfrac{\beta(\|x_{k-1}-Tx_k\|)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$, $B_\gamma:=\dfrac{\gamma(\|x_{k-1}-Tx_k\|)\|x_k-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$,
$C_\delta:=\dfrac{\delta(\|x_{k-1}-Tx_k\|)\|x_{k-1}-Tx_k\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$; $D_\epsilon:=\dfrac{\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$.
So that, $A_\beta:=\dfrac{\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$, $B_\gamma:=\dfrac{\gamma(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$,
$C_\delta:=\dfrac{\delta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_{k+1}\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$; $D_\epsilon:=\dfrac{\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$.
Thus, $A_\beta:=\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|$, $B_\gamma:=\gamma(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|$,
$C_\delta:=\delta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_{k+1}\|$; $D_\epsilon:=\dfrac{\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}=0$.
Then the conclusion follows from proof of Theorem 4. □

Corollary 7. *Let $(X,\|\cdot\|)$ be a Banach space and $T:X\to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\}\in\mathcal{F}_{Ger}$*

satisfying;

$$\|Tx-Ty\|^2 \leq \alpha(\|x-Ty\|)\|x-y\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^{A_\beta^{B_\gamma^{D_\epsilon^{C_\delta}}}}}}}} \quad ; \tag{29}$$

where, $A_\beta := \dfrac{\beta(\|x-Ty\|)\|x-Tx\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-Ty\|)\|y-Ty\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$,

$C_\delta := \dfrac{\delta(\|x-Ty\|)\|x-Ty\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-Ty\|)\|y-Tx\|}{1+\epsilon(\|x-Ty\|)\|y-Tx\|}$, *for all distinct* $x,y \in X$ *and* $\ln(\|x-y\|) \leq \|x-y\|$, *so that there exists* $\theta > 0$, $\|x \tilde{-} y\| = \frac{\|x-y\|}{\theta}$ *for which* $\theta \geq \sup_{x,y\in X}\|x-y\|$, $\|x \tilde{-} y\|^k \leq \|x \tilde{-} y\|$ *for all* $k \in \mathbb{N}$. *Then the sequence* $\{x_k\}_{k\in\mathbb{N}}$ *defined by* $Tx_{k-1} = x_k$ *converges to a unique fixed point of* T *in* X.

Proof. Suppose, if possible, otherwise. Let $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $\|x_k - x_{k+1}\| > 0$, so that $\|x_k - x_{k+1}\| = \|Tx_{k-1} - Tx_k\|$. From inequality (29), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$\|x_k - x_{k+1}\|^2 = \|Tx_{k-1} - Tx_k\|^2$$

$$\leq \alpha(\|x_{k-1} - Tx_k\|)\|x_{k-1} - x_k\|^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^{A_\beta^{B_\gamma^{D_\epsilon^{C_\delta}}}}}}}} \quad ; \tag{30}$$

where;

$A_\beta := \dfrac{\beta(\|x_{k-1}-Tx_k\|)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$, $B_\gamma := \dfrac{\gamma(\|x_{k-1}-Tx_k\|)\|x_k-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$,

$C_\delta := \dfrac{\delta(\|x_{k-1}-Tx_k\|)\|x_{k-1}-Tx_k\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$; $D_\epsilon := \dfrac{\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-Tx_k\|)\|x_k-Tx_{k-1}\|}$.

So that, $A_\beta := \dfrac{\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$, $B_\gamma := \dfrac{\gamma(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$,

$C_\delta := \dfrac{\delta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_{k+1}\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$; $D_\epsilon := \dfrac{\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}$.

Thus, $A_\beta := \beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|$, $B_\gamma := \gamma(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|$,

$C_\delta := \delta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_{k+1}\|$; $D_\epsilon := \dfrac{\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|}{1+\epsilon(\|x_{k-1}-x_{k+1}\|)\|x_k-x_k\|} = 0$.

Therefore we have by taking $W_{\|\Delta x_k\|} := \|x_k - x_{k+1}\|^2$, thus inequality (30) reduced to

$$W_{\|\Delta x_k\|} \leq \alpha(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|^{A_\beta^{B_\gamma}}$$

$$\leq \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\left(A_\beta^{B_\gamma}\ln(\|x_k-x_{k-1}\|)\right)^j}{j!}$$

$$= \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{A_\beta^{jB_\gamma}\ln^j(\|x_k-x_{k-1}\|)}{j!}$$

$$
\begin{aligned}
&\leq \alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{A_{\beta}^{jB_{\gamma}}\|x_k-x_{k-1}\|}{j!}\\
&=\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\exp\left(jB_{\gamma}\ln(A_{\beta})\right)\|x_k-x_{k-1}\|}{j!}\\
&=\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\|x_k-x_{k-1}\|}{j!}\sum_{r=0}^{\infty}\frac{\left(jB_{\gamma}\ln(A_{\beta})\right)^r}{r!}\\
&=\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\|x_k-x_{k-1}\|}{j!}\sum_{r=0}^{\infty}\frac{j^r B_{\gamma}^r\ln^r(A_{\beta})}{r!}\\
&=\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\frac{\|x_k-x_{k-1}\|}{j!}\sum_{r=0}^{\infty}\frac{j^r\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|^r\times}{r!}\\
&\qquad\qquad\frac{\ln^r(\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|)}{r!}\\
&=\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|^r\times}{j!r!}\\
&\qquad\qquad\ln^r(\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|)\|x_k-x_{k-1}\|\\
&\leq\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|^r\\
&\qquad\ln^r(\beta(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|)\|x_k-x_{k-1}\|\\
&\leq\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|\\
&\qquad\beta^r(\|x_{k-1}-x_{k+1}\|)\|x_{k-1}-x_k\|\|x_k-x_{k-1}\|\\
&\leq\alpha(\|x_{k-1}-x_{k+1}\|)\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|\,\beta^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k-1}\|\\
&=\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k+1}\|\|x_k-x_{k-1}\|.
\end{aligned}
$$

Hence;

$$
\begin{aligned}
\|x_k-x_{k+1}\|&\leq\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k-1}\|\\
&=\left(\sum_{j=0}^{\infty}\frac{j^r}{j!r!}\frac{\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)}{1-\gamma^r(\|x_{k-1}-x_{k+1}\|)}\right)\|x_k-x_{k-1}\|. \qquad (31)
\end{aligned}
$$

Therefore, from inequality (31), we get;

$$
\|x_k-x_{k+1}\|\leq\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k-x_{k-1}\|. \qquad (32)
$$

Therefore,

$$\begin{aligned}\|x_k - x_{k+1}\| &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k - x_{k-1}\| \\ &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-2}-x_k\|)\beta^r(\|x_{k-2}-x_k\|)\gamma^r(\|x_{k-2}-x_k\|)\|x_{k-1} - x_{k-2}\| \\ &\leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-3}-x_{k-1}\|)\beta^r(\|x_{k-3}-x_{k-1}\|)\gamma^r(\|x_{k-3}-x_{k-1}\|)\|x_{k-2} - x_{k-3}\| \\ &\vdots \\ &\leq \|x_0 - x_1\|. \end{aligned} \tag{33}$$

Thus, $\{\|x_k - x_{k+1}\|\}_{k\in\mathbb{N}}$ is a non-increasing sequence and bounded below for which the sequence converges to some real number $a \geq 0$ (say). If $a > 0$, then we see that

$$\|x_k - x_{k+1}\| \leq \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\|x_k - x_{k-1}\|. \tag{34}$$

Thus,

$$1 \leq \lim_{k\to\infty}\inf\left\{\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{j^r}{j!r!}\alpha(\|x_{k-1}-x_{k+1}\|)\beta^r(\|x_{k-1}-x_{k+1}\|)\gamma^r(\|x_{k-1}-x_{k+1}\|)\right\}. \tag{35}$$

This implies that $\lim_{k\to\infty}\|x_{k-1} - x_{k+1}\| = 0$, which is a contradiction. Therefore,

$$\lim_{k\to\infty}\|x_k - x_{k+1}\| = 0. \tag{36}$$

To show that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X follows with little modification from inequality (11). Therefore, we can see that the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X. □

Proposition 8. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ satisfying;*

$$\|Tx - Ty\| \leq \alpha(\|x-y\|)\|x-y\|^{A_\beta^{D_\epsilon^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}}; \tag{37}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, *for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.*

Remark 9. The $\cdot^{\cdot^{\cdot}}$ in the upper most of inequality (37) means that $A_\beta^{D_\epsilon^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}$ is climbing upstairs

simultaneously to infinity.

Proof. Suppose that $\{x_k\}_{k\in\mathbb{N}} \subseteq X$, we put $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $\|x_k - x_{k+1}\| > 0$, so that $\|x_k - x_{k+1}\| = \|Tx_{k-1} - Tx_k\|$. From inequality (37), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$\|Tx_{k-1} - Tx_k\| \leq \alpha(\|x_{k-1} - x_k\|)\|x_{k-1} - x_k\|^{A_\beta^{D_\epsilon^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}}; \tag{38}$$

where,

$A_\beta := \dfrac{\beta(\|x_{k-1} - x_k\|)\|x_{k-1} - Tx_{k-1}\|}{1 + \epsilon(\|x_{k-1} - x_k\|)\|x_k - Tx_{k-1}\|}$, $B_\gamma := \dfrac{\gamma(\|x_{k-1} - x_k\|)\|x_k - Tx_k\|}{1 + \epsilon(\|x_{k-1} - x_k\|)\|x_k - Tx_{k-1}\|}$,

$C_\delta := \dfrac{\delta(\|x_{k-1} - x_k\|)\|x_{k-1} - Tx_k\|}{1 + \epsilon(\|x_{k-1} - x_k\|)\|x_k - Tx_{k-1}\|}$; $D_\epsilon := \dfrac{\epsilon(\|x_{k-1} - x_k\|)\|x_k - Tx_{k-1}\|}{1 + \epsilon(\|x_{k-1} - x_k\|)\|x_k - Tx_{k-1}\|}$.

Thus,

$A_\beta := \beta(\|x_{k-1} - x_k\|)\|x_{k-1} - x_k\|$, $B_\gamma := \gamma(\|x_{k-1} - x_k\|)\|x_k - x_{k+1}\|$,

$C_\delta := \delta(\|x_{k-1} - x_k\|)\|x_{k-1} - x_{k+1}\|$; $D_\epsilon := \epsilon(\|x_{k-1} - x_k\|)\|x_k - x_k\|$.

So inequality (38) becomes;

$$\|Tx_{k-1} - Tx_k\| \leq \alpha(\|x_{k-1} - x_k\|)\|x_{k-1} - x_k\|. \tag{39}$$

We need only to show that the existence of such an α in $\mathcal{F}_{Ger}$ is equivalent to say that for $\|x_k\| \to \|x^*\|$ in X, which means that $x_k \to x^*$ strongly with x^* a unique fixed point of T, iff for any two subsequences x_{h_n} and x_{h_k} of x_k with $x_{h_n} \neq x_{h_k}$, we have that $\Omega_k \to 1$ only if $\|\cdot\|_k \to 0$,i.e., $\|x_k - y_k\| \leq \|x_k - x^*\| + \|y_k - x^*\| \to 0$, where we take for any pair of sequences x_k and y_k with $x_k \neq y_k$, we write $\|\cdot\|_k = \|x_k - y_k\|$ and $\Omega_k = \dfrac{\|T(x_k) - T(y_k)\|}{\|x_k - y_k\|}$. First assume such an α exists. Let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be two subsequences with $x_{h_n} \neq x_{k_n}$. Assume that $\Omega_k \to 1$ as $k \to \infty$. Then it follows immediately from inequality (39) that $\alpha(\|x_{h_n} - x_{k_n}\|) \to 1$ as $n \to \infty$. But then since $\alpha \in \mathcal{F}_{Ger}$, we have $\|x_{h_n} - x_{k_n}\| \to 0$. Next assume that the sequential condition holds. Define $\alpha : \mathbb{R}^+ \to \mathbb{R}$ as follows $\alpha(t) = \sup\left\{\frac{\|T(x_n) - T(x_m)\|}{\|x_n - x_m\|} : \|x_n - x_m\| \geq t\right\}$. From inequality (39), the quotients are all below 1 and so α is defined for all $t > 0$ and $\alpha \leq 1$. Now assume that $\alpha(t_n) \to 1$ for $t_n \in \mathbb{R}^+$. We may further assume without loss of generality that $1 - \dfrac{1}{n} \leq \alpha(t_n) \leq 1$. We must show that $t_n \to 0$. But $\alpha(t_n)$ is the above least upper bound. So there is for each $n > 0$ a pair $\{x_{h_n}\}$ and $\{x_{k_n}\}$ of $\{x_k\}$ with $\|x_{h_n} - x_{k_n}\| \geq t_n$ and $1 - \dfrac{1}{n} < \dfrac{\|T(x_{h_n}) - T(x_{k_n})\|}{\|x_{h_n} - x_{k_n}\|} \leq \alpha(t_n)$. So the sequence $\Omega_k \to 1$ as $k \to \infty$. But then by the sequential condition above, $\|x_{h_n} - x_{k_n}\| \leq \|x_{h_n} - x^*\| + \|x^* - x_{k_n}\| \to 0$. So $t_n \to 0$, as was shown. This completes the proof of Proposition 8. □

Proposition 10. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in$*

$\mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;

$$\|Tx-Ty\| \lneqq \alpha(\|x-y\|)\|x-y\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}}}}}} \quad ; \tag{40}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, *for all distinct* $x,y \in X$ *for all* $k \in \mathbb{N}$. *Then the sequence* $\{x_k\}_{k\in\mathbb{N}}$ *defined by* $Tx_{k-1} = x_k$ *converges to a unique fixed point of* T *in* X.

Proof. We give the proof of Proposition 10 in its full strength. The proof decomposed into two cases:

Case (a) Observe first that if $\theta = 0$, then inequality (40) reduced to

$$\|Tx-Ty\| < \alpha(\|x-y\|)\|x-y\|\,. \tag{41}$$

First assume that $\|x_n\| \to \|x^*\|$ in X which means that $x_n \to x^*$ strongly and let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be any two subsequences of $\{x_h\}$. Then clearly $\|x_{h_n} - y_{h_n}\| \to 0$, so the condition is satisfied. Next assume the condition is satisfied for a given initial point x_0 in X. Then $\{\|x_n - y_n\|\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative real numbers and so approaches some $\epsilon^* \geq 0$. Assume that $\epsilon^* > 0$. Then letting $h_n = n$ and $k_n = n+1$, we have that $\|x_n - y_n\| \to \epsilon^* > 0$ while $\Omega_n \to 1$ as $n \to \infty$. So the condition is contradicted. Thus $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$. This implies that $\|x_n - x_{n+1}\| \leq \|x_n - x^*\| + \|x^* - x_{n+1}\| \to 0$. Now assume the given sequence of iterates $\{x_n\}$ is not Cauchy. Then there exists some $\epsilon^* > 0$ such that every tail $\{x_n\}_{n\geq N}$ of the sequence has diameter $D_N = \sup_{n,m\geq N}\|x_n - x_m\| > \epsilon^*$. Given this ϵ^*, we will construct a pair of subsequences violating the condition. For any $n > 0$, let N_n be large that $\|x_m - x_{m+1}\| < \frac{1}{n}$ for all $m \geq N_n$, as is possible that $\|x_m - x_{m+1}\| \to 0$. Let $h_n \geq N_n$ be the lowest integer such that for some $k_n > h_n$, $\|x_{h_n} - x_{h_k}\| > \epsilon^*$. Such pairs exist by the above diameter condition. Next choose k_n to be the least such integer above h_n. Then either $k_n - 1 = h_n$ or $\|x_{h_n} - x_{h_k-1}\| \leq \epsilon^*$. In either case we have $\epsilon^* \leq \|x_{h_n} - x_{h_k-1}\| < \epsilon^* + \frac{1}{n}$. Moreover, using the triangular inequality, from inequality (41), we have

$$1 \geq \Omega_n = \frac{\|T(x_{h_n}) - T(y_{k_n})\|}{\|x_n - y_n\|} \geq \frac{\|x_{h_n} - x_{h_k-1}\| - \frac{2}{n}}{\|x_{h_n} - x_{h_k-1}\|}. \tag{42}$$

So $\Omega_n \to 1$ while $\|x_{h_n} - x_{h_k-1}\| \to \epsilon^*$, again violating the condition. So $\{x_n\}$ must be a Cauchy sequence in X and as X is complete, we have $x_n \to x^*$ for some x^* in X. Then by the usual arguments, x^* is a unique fixed point of T and the proof of (a) is complete.

Case (b) if $\theta \neq 0$, then inequality (40) becomes;

$$\|Tx-Ty\| \lneqq \alpha(\|x-y\|)\|x-y\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}}}}}}; \tag{43}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, for all distinct $x,y \in X$ for all $k \in \mathbb{N}$. Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $\|x_k - x_{k+1}\| > 0$, so that $\|x_k - x_{k+1}\| = \|Tx_{k-1} - Tx_k\|$. From inequality (43), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$\|Tx_{k-1}-Tx_k\| \lneqq \alpha(\|x_{k-1}-x_k\|)\|x_{k-1}-x_k\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}}}}}}; \tag{44}$$

where,
$A_\beta := \dfrac{\beta(\|x_{k-1}-x_k\|)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$, $B_\gamma := \dfrac{\gamma(\|x_{k-1}-x_k\|)\|x_k-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$,
$C_\delta := \dfrac{\delta(\|x_{k-1}-x_k\|)\|x_{k-1}-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$; $D_\epsilon := \dfrac{\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$.
Thus,
$A_\beta := \beta(\|x_{k-1}-x_k\|)\|x_{k-1}-x_k\|$, $B_\gamma := \gamma(\|x_{k-1}-x_k\|)\|x_k-x_{k+1}\|$,
$C_\delta := \delta(\|x_{k-1}-x_k\|)\|x_{k-1}-x_{k+1}\|$; $D_\epsilon := \epsilon(\|x_{k-1}-x_k\|)\|x_k-x_k\|$.
So inequality (44) becomes;

$$\|Tx_{k-1}-Tx_k\| \lneqq \alpha(\|x_{k-1}-x_k\|)\|x_{k-1}-x_k\|. \tag{45}$$

We need only to show that the existence of such an α in $\mathcal{F}_{Ger}$ is equivalent to say that for $\|x_n\| \to \|x^*\|$ in X, with x^* a unique fixed point of T, iff for any two subsequences x_{h_n} and x_{h_k} of x_n with $x_{h_n} \neq x_{h_k}$, we have that $\Omega_n \to 1$ only if $\|x_n - y_n\| \to 0$, where we take for any pair of sequences x_n and y_n with $x_n \neq y_n$, were $\Omega_n = \dfrac{\|T(x_n)-T(y_n)\|}{\|x_n-y_n\|}$. First assume such an α exists. Let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be two subsequences with $x_{h_n} \neq x_{k_n}$. Assume that $\Omega_n \to 1$ as $n \to \infty$. Then it follows from inequality (45) that $\alpha(\|x_{h_n} - x_{k_n}\|) \to 1$. But then since $\alpha \in \mathcal{F}_{Ger}$, we have $\|x_{h_n} - x_{k_n}\| \to 0$. Next assume that the sequential condition holds. Define $\alpha : \mathbb{R}^+ \to \mathbb{R}$ as follows $\alpha(t) = \sup\left\{\dfrac{\|T(x_n)-T(x_m)\|}{\|x_n-x_m\|} : \|x_n - x_m\| \geq t\right\}$. From inequality (45) the quotients are all below 1 and so α is defined for all $t > 0$ and $\alpha < 1$. Now assume that $\alpha(t_n) \to 1$ for $t_n \in \mathbb{R}^+$. We may further assume without loss of generality that $1 - \dfrac{1}{n} \leq \alpha(t_n) < 1$. We must show $t_n \to 0$. But $\alpha(t_n)$ is the above least upper bound. So there is for each $n > 0$ a pair $\{x_{h_n}\}$ and $\{x_{k_n}\}$ of $\{x_n\}$ with

$\|x_{h_n} - x_{k_n}\| \geq t_n$ and

$$1 - \frac{1}{n} < \frac{\|T(x_{h_n}) - T(x_{k_n})\|}{\|x_{h_n} - x_{k_n}\|} < \alpha(t_n). \tag{46}$$

So the sequence $\Omega_n \to 1$ as $n \to \infty$. But then by the sequential condition above, $\|x_{h_n} - x_{k_n}\| \to 0$. So $t_n \to 0$, as was shown. This completes the proof of Proposition 10. □

Corollary 11. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;*

$$\|Tx - Ty\| \leq \alpha(\|x-y\|)\|x-y\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}; \tag{47}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, *for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.*

Proof. We prove as follows;
(a) if $\theta = 0$, then inequality (47) becomes;

$$\|Tx - Ty\| \leq \alpha(\|x-y\|)\|x-y\|. \tag{48}$$

This is Geraghty contraction in Banach space and the proof follows from small modification see Geraghty (1973).
(b) if $\theta \neq 0$, then inequality (47) becomes;

$$\|Tx - Ty\| \leq \alpha(\|x-y\|)\|x-y\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}; \tag{49}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $\|x_k - x_{k+1}\| > 0$, so that $\|x_k - x_{k+1}\| = \|Tx_{k-1} - Tx_k\|$. From inequality (49), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$\|Tx_{k-1} - Tx_k\| \leq \alpha(\|x_{k-1} - x_k\|)\|x_{k-1} - x_k\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}; \tag{50}$$

where,
$A_\beta := \dfrac{\beta(\|x_{k-1}-x_k\|)\|x_{k-1}-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$, $B_\gamma := \dfrac{\gamma(\|x_{k-1}-x_k\|)\|x_k-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$,
$C_\delta := \dfrac{\delta(\|x_{k-1}-x_k\|)\|x_{k-1}-Tx_k\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$; $D_\epsilon := \dfrac{\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}{1+\epsilon(\|x_{k-1}-x_k\|)\|x_k-Tx_{k-1}\|}$.
Thus,

$A_\beta := \beta(\|x_{k-1}-x_k\|)\|x_{k-1}-x_k\|$, $B_\gamma := \gamma(\|x_{k-1}-x_k\|)\|x_k-x_{k+1}\|$,
$C_\delta := \delta(\|x_{k-1}-x_k\|)\|x_{k-1}-x_{k+1}\|$; $D_\epsilon := \epsilon(\|x_{k-1}-x_k\|)\|x_k-x_k\|$.
So inequality (49) becomes;

$$\|Tx_{k-1}-Tx_k\| \leq \alpha(\|x_{k-1}-x_k\|)\|x_{k-1}-x_k\|. \tag{51}$$

It follows Corollary 8 above. Therefore, the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1}=x_k$ converges to a unique fixed point of T in X. □

Proposition 12. *Let $(X,\|\cdot\|)$ be a Banach space and $T: X \to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;*

$$\|Tx-Ty\| \leq \alpha(\|x-y\|)\|x-y\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{C_\delta^{\cdot^{\cdot}}}}}}}; \tag{52}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, *for all distinct $x,y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1}=x_k$ converges to a unique fixed point of T in X.*

Proof. It follows from the proof of Corollary 11 and we conclude that the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1}=x_k$ converges to a unique fixed point of T in X. □

Proposition 13. *Let $(X,\|\cdot\|)$ be a Banach space and $T: X \to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;*

$$\|Tx-Ty\| < \alpha(\|x-y\|)\|x-y\|^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{\cdot^{\cdot^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{\cdot^{\cdot}}}}}}}}}}}}; \tag{53}$$

where, $A_\beta := \dfrac{\beta(\|x-y\|)\|x-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, $B_\gamma := \dfrac{\gamma(\|x-y\|)\|y-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$,
$C_\delta := \dfrac{\delta(\|x-y\|)\|x-Ty\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$; $D_\epsilon := \dfrac{\epsilon(\|x-y\|)\|y-Tx\|}{1+\epsilon(\|x-y\|)\|y-Tx\|}$, *for all distinct $x,y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1}=x_k$ converges to a unique fixed point of T in X.*

Proof. We give the proof of Proposition 13 in its full strength. We prove in two cases:
(a) Observe first that if $\theta = 0$, then inequality (53) reduced to

$$\|Tx-Ty\| < \alpha(\|x-y\|)\|x-y\|. \tag{54}$$

First assume that $\|x_n\| \to \|x^*\|$ in X which means that $x_n \to x^*$ strongly and let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be any two subsequences of $\{x_h\}$. Then clearly $\|x_{h_n}-y_{h_n}\| \to 0$, so the condition is satisfied. Next

assume the condition is satisfied for a given initial point x_0 in X. Then $\{\|x_n - y_n\|\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative real numbers and so approaches some $\kappa \geq 0$. Assume $\kappa > 0$. Then letting $h_n = n$ and $k_n = n+1$, we have that $\|x_n - y_n\| \to \kappa > 0$ while $\Omega_n \to 1$ as $n \to \infty$. So the condition is violated. Thus $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$. This implies that $\|x_n - x_{n+1}\| \leq \|x_n - x^*\| + \|x^* - x_{n+1}\| \to 0$. Now assume the given sequence of iterates $\{x_n\}$ is not Cauchy. Then there exists some $\kappa > 0$ such that every tail $\{x_n\}_{n\geq N}$ of the sequence has diameter $G_N = \sup_{n,m\geq N}\|x_n - x_m\| > \kappa$. Given this κ, we will construct a pair of subsequences violating the condition. For any $n > 0$, let N_n be large that $\|x_m - x_{m+1}\| < \frac{1}{n}$ for all $m \geq N_n$, as is possible that $\|x_m - x_{m+1}\| \to 0$. Let $h_n \geq N_n$ be the lowest integer such that for some $k_n > h_n$, $\|x_{h_n} - x_{h_k}\| > \kappa$. Such pairs exist by the above diameter condition. Next choose k_n to be the least such integer above h_n. Then either $k_n - 1 = h_n$ or $\|x_{h_n} - x_{h_k-1}\| \leq \kappa$. In either case we have $\kappa \leq \|x_{h_n} - x_{h_k-1}\| < \kappa + \frac{1}{n}$. Moreover, using the triangular inequality, from inequality (54), we have

$$1 \geq \Omega_n = \frac{\|T(x_{h_n}) - T(y_{k_n})\|}{\|x_n - y_n\|} \geq \frac{\|x_{h_n} - x_{h_k-1}\| - \frac{2}{n}}{\|x_{h_n} - x_{h_k-1}\|}. \tag{55}$$

So $\Omega_n \to 1$ while $\|x_{h_n} - x_{h_k-1}\| \to \kappa$, again violating the condition. Therefore, $\{x_n\}$ must be a Cauchy sequence in X and as X is complete, we have $x_n \to x^*$ for some x^* in X. Then by the usual arguments, x^* is a unique fixed point of T and the proof of (a) is complete.
(b) The proof follows from proof of part (b) of Proposition 10 and we are done. □

4 Conclusion

In this paper, we established novel conditions for the existence and uniqueness of fixed points for a newly introduced class of generalized tower mappings in Banach spaces. By integrating Geraghty auxiliary functions with multi-layered, rational exponential bounds, we constructed a theoretical framework that systematically relaxes the rigid constraints of classical contraction principles. Consequently, our findings generalize, improve, and unify existing fixed point theorems in literature.

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